4 An Alternative Description of the Cartesian Plane

(4.1) Definition

If $A = (x_1, y_1), B = (x_2, y_2) \in \mathbb{R}^2$ and $r \in \mathbb{R}$ then

(i) $A + B = (x_1 + x_2, y_1 + y_2) \in \mathbb{R}^2$ (ii) $rA = (rx_1, ry_1) \in \mathbb{R}^2$
(iii) $A - B = A + (-1)B = (x_1 - x_2, y_1 - y_2) \in \mathbb{R}^2$
(iv) $\langle A, B\rangle = x_1x_2 + y_1y_2 \in \mathbb{R}$ (v) $\|A\| = \sqrt{\langle A, A\rangle} \in \mathbb{R}$

(4.2) Proposition

For all $A, B, C \in \mathbb{R}^2$ and $r, s \in \mathbb{R}$

(i) $A + B = B + A$ (ii) $(A + B) + C = A + (B + C)$ (iii) $r(A + B) = rA + rB$
(iv) $(r + s)A = rA + sA$ (v) $\langle A, B\rangle = \langle B, A\rangle$ (vi) $\langle rA, B\rangle = r\langle A, B\rangle$
(vii) $\langle A + B, C\rangle = \langle A, C\rangle + \langle B, C\rangle$ (viii) $\|rA\| = |r|\|A\|$ (ix) $\|A\| > 0$ *if* $A \neq (0, 0)$

(4.3) Definition (L_{AB})

Define the line L_{AB} through the distinct points A and B by $L_{AB} = \{X \in \mathbb{R}^2 \mid X = A + t(B - A),\ t \in \mathbb{R}\}$.

1. Prove Proposition 4.2.

2. Find point $C \in \mathbb{R}^2$ which does not belong to line L_{AB}.

3. Let $x = a$ be a given line (for some $a \in \mathbb{R}$). Is it possible to find two points A and B such that points from $L_{AB} = \{X \in \mathbb{R}^2 \mid X = A + t(B - A),\ t \in \mathbb{R}\}$ are the same as points from given line $x = a$? Find such two points.

4. Let $y = kx + n$ be a given line (for some $k, n \in \mathbb{R}$, $k \neq 0$, $n \neq 0$). Is it possible to find two points A and B such that points from $L_{AB} = \{X \in \mathbb{R}^2 \mid X = A + t(B - A),\ t \in \mathbb{R}\}$ are the same as points from given line $y = kx + n$? Find such two points.

5. Let L_{AB} be given line, where $A = (1, 2)$, $B = (3, 0)$. Find for which real numbers k and n line L_{AB} is equal to $y = kx + n$.

6. Let $\mathcal{L}'$ denote the collection of all subsets of $\mathbb{R}^2$ of the form L_{AB}. Show that then $\{\mathbb{R}^2, \mathcal{L}'\}$ is the Cartesian Plane (and hence is an incidence geometry).

7. Let $\mathcal{L}'$ denote the collection of all subsets of $\mathbb{R}^2$ of the form L_{AB}. Prove directly that $\{\mathbb{R}^2, \mathcal{L}'\}$ is an incidence geometry without any reference to the initial model $\{\mathbb{R}^2, \mathcal{L}'\}$.

8. Show that if $A, B \in \mathbb{R}^2$ then $d_E(A, B) = \|A - B\|$.

9. Let L_{AB} denote given line, and let $f : L_{AB} \to \mathbb{R}$ denote function defined by $f(A + t(B - A)) = t\|A - B\|$. Show that (i) f is injection (ii) f satisfy Ruler Equation (for model $\{\mathbb{R}^2, \mathcal{L}_E, d_E\}$).

10. Show that if L_{AB} is a Cartesian line then $f : L_{AB} \to \mathbb{R}$ defined by $f(A + t(B - A)) = t\|A - B\|$ is a ruler for $\{\mathbb{R}^2, \mathcal{L}_E, d_E\}$.

(4.4) Proposition (Cauchy-Schwarz Inequality)

If $X, Y \in \mathbb{R}^2$ then $|\langle X, Y\rangle| \leq \|X\| \cdot \|Y\|$. Furthermore, equality holds if and only if either $Y = (0, 0)$ or $X = tY$ for some $t \in R$.

11. Prove Cauchy-Schwarz Inequality (see Proposition 4.4).

(4.5) Definition (triangle inequality)

A distance function d on $\mathcal{S}$ satisfies the triangle inequality if $d(A, C) \leq d(A, B) + d(B, C)$ for all $A, B, C \in \mathcal{S}$.

12. Prove that the Euclidean distance function d_E satisfies the triangle inequality.

13. Prove that the Taxicab distance d_T satisfies the triangle inequality.

14. Prove that the max distance d_S on $\mathbb{R}^2$ satisfies the triangle inequality.

5 Betweenness

(5.1) Definition (B is between A and C)
B is between A and C if A, B, and C are distinct collinear points in the metric geometry $\{\mathcal{S}, \mathcal{L}, d\}$ and if $d(A,B) + d(B,C) = d(A,C)$.

1. In the Taxicab Plane, find three points A, B, C which are not collinear but $d_T(A,C) = d_T(A,B) + d_T(B,C)$. This problem shows why the definition of between requires collinear points.

Because we will be using betweenness and distance constantly throughout the rest of all problems we adopt the following simplified notation.

Notation. In a metric geometry $\{\mathcal{S}, \mathcal{L}, d\}$ (i) $A-B-C$ means B is between A and C (ii) AB denotes the distance $d(A,B)$.

2. Let $A(4,4)$, $B(1,5)$, and $C(5,3)$ in the Poincaré plane. Show that $B-A-C$.

3. If $A(4,7)$, $B(1,1)$, and $C(2,3)$ prove that $A-C-B$ in the Taxicab Plane.

4. Let $A = (-1/2, \sqrt{3}/2)$, $B(0,1)$, and $C(1/2, \sqrt{3}/2)$ denote three points in the Poincaré plane. Show that then $A-B-C$.

5. Show that if $A-B-C$ then $C-B-A$.

6. Let p denote a line, let f denote coordinate system for line p, and let A,B,C denote three different points on p, with coordinates x,y,z, respectively. Show that if $A-B-C$ then it is not possible non of the following: (1°) $z<x<y$ (2°) $x<z<y$ (3°) $y<z<x$.

7. Let $A(x_1,y_1)$, $B(x_2,y_2)$, and $C(x,y)$ be three collinear points in the Euclidean Plane with $x_1 < x_2$. Prove that $A-C-B$ if and only if $x_1 < x < x_2$.

8. With respect to Problem 7, formulate and prove a condition for $A-C-B$ if A and B are on the same type I line in the Poincaré Plane.

(5.2) Definition ($x*y*z$)
If x, y, and z are real numbers, then y is between x and z (written $x*y*z$) if either $x<y<z$ or $z<y<x$.

(5.3) Theorem ($A-B-C$ if and only if $x*y*z$)
Let ℓ be a line and f a coordinate system for ℓ. If A, B, and C are three points of ℓ with coordinates x, y, and z respectively, then $A-B-C$ if and only if $x*y*z$.

9. Prove theorem above.

10. For given three distinct points on a line, show that one and only one of these points is between the other two.

11. Show that in the Euclidean plane $A-B-C$ if and only if there is a number t with $0<t<1$ and $B = A + t(C-A)$.

12. Show that if A and B are distinct points in a metric geometry then (i) there is a point C with $A-B-C$; and (ii) there is a point D with $A-D-B$.

(5.4) Definition ($A-B-C-D$)
$A-B-C-D$ means that $A-B-C$, $A-B-D$, $A-C-D$, and $B-C-D$.

13. Show that $A-B-C-D$ in a metric geometry implies that $\{A,B,C,D\}$ is a collinear set.

14. Suppose that $A-B-C$ and $B-C-D$ in a metric geometry. Show that then $A-B-D$, and $A-C-D$, and hence that $A-B-C-D$.

15. Let A,B,C,D be distinct collinear points in a metric geometry. Show that the points can always be named in order so that $A-B-C-D$.